Computational and Analytical Approaches in Solving Problems Requiring Mathematical Investigation

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Theoretical Framework

Quasi-empirical nature of mathematics.

"Deductivist style hides the struggle, hides the adventure" (Lakatos, 1976, p.142).

It is a common approach among students to use empirical means or just check a number of examples in order to prove a general statement instead of constructing a deductive proof (Balacheff, 1988; Harel and Sowder, 1998; Weber, 2001).

Instructors in proof-oriented courses warn students not to "prove by example."

However, mathematicians emphasize that using examples plays an important role in conjecturing and proving (Thurston, 1995; Alcock, 2004).

How do students use examples in solving problems by investigation?

Methodology

Participants: Undergraduate students enrolled in the course MAST 217 "Introduction to Mathematical Thinking"

Data: Written responses to homework assignments

Data analysis: Data coded by using the "Cognitive Processes in Mathematical Investigation" model (Yeo, 2017)

Method: Triangulation

PHASE	STAGE
Entry	**Understanding the Task**
	Problem Posing
Attack	**Specializing and Using Other Heuristics**
	Conjecturing
	Justifying
	Generalizing
Review	Checking
	Extension

Results

The haggling problem

John is trying to sell Mark a bike for **a** dollars.

Mark does not agree on the price and offers **b** dollars (0< b < a).

John does not agree on this price but comes down to **(a + b)/2 = 1/2 a + 1/2 b**.

Mark responds by offering **(b + (a + b)/2)/2 = 1/4 a + 3/4 b**.

They continue haggling this way, each time taking the average of the previous two amounts.

On what amount will they converge? Express the amount in terms of a and b.

Explain your reasoning and justify your response.

Have you tried to verify your answer? If yes, how?

Two types of ways of using examples to attack the haggling problem were identified in students' solutions:

Computational: arithmetical exploration of many concrete amounts or coefficients (usually by using a computer), followed by conjecturing

Analytical: algebraical exploration of the sequence of coefficients, searching for patterns and then using other heuristics, such as analogy, deduction, structure recognition, algebraic manipulations.

Computational approach (7 solutions)

3 successful

4 unsuccessful

Student #27

They will converge on the amount of a/3+(2/3)b which is close to 0.333a+0.666b

ANALYSIS:

John will offer for a dollars

Mark will offer for b dollars, and we know 0< b < a

Then John will offer for (a+b)/2=0.5a+0.5b

Mark will offer for (a+3b)/4=0.25a+0.75b

……….

John will offer for (171a+341b)/512=0.3339a+0.6660b

Mark will offer for (341a+683b)/1024=0.3330a+0.6669b

We can see easily in analysis above that the amount is converging to a/3+(2/3)b which is close to 0.333a+0.666b

Specially in the last two offers you can see that 0.333a and 0.666b repeating and they will continue to repeat for all next offers

I verified my answer with some examples by trying with different numbers, they all confirmed that they will converge to a/3+(2/3)b

Let a=$100, b=$50, we see that 0< b < a

John will offer $100

Mark will offer $50

then John will offer $75

Mark will offer $62.50

…………

John will offer $66.64

Mark will offer $66.66

As we see in example above, it converges to amount of $66.66, therefore a/3+(2/3)b=(100/3)+(100/3)=$66.66... This example proves that they will converge on the amount of a/3+(2/3)b

Analytical approach (19 solutions)

8 successful

11 unsuccessful

Student #10

We aren't given the values of a and b, but we do know that the first value, a is the highest term in the sequence and the second term, b, is the lowest. We also know the amount they will converge on is somewhere between a and b. If we assume a to be 0 and b to be 1 we'll converge on a number between 0 and 1. If we call that number x and convert it to % then the final amount will be x% of the way from b (lowest term) to a (highest term).

This can be represented as b + x (a-b)

n1 = 1 (our chosen value for a)

n2 = 0 (our chosen value for b)

n3 = ½ (the average of the last two terms)

n4 = ¼ (the average of the last two terms)

n5 = 3/8 (again the average of the last two terms)

If for every n we look at the difference between n and n-1, we get a pattern with the first few terms as:

n1 = 1

n2 = -1

n3 = +1/2

n4 = -1/4

n5 = 1/8

In each case the next n is equal to the previous n multiplied by -1/2. Taking the sum of all n from n=1 to n=infinity we can find what value of x should be used in the final answer b + x (a-b).

Starting from n2, we have a geometric series with a = -1 and q = -1/2.

For a geometric series of this time the sum of the term from n=2 to n=infinity can be represented as $a(\frac{1}{1-q})$. In this case we get -1(2/3) = -2/3. But this doesn't include n1, which has a value of 1. So the sum from n=1 to n=infinity = 1 – 2/3 = 1/3. Thus the value for x in the equation b + x (a-b) is 1/3.

So the final equation is: $b + \frac{a-b}{3}$

This can be verified using Microsoft Excel. I first entered arbitrary values for a and b in cells A1 and A2 respectively. In cell A3 I used the formula =AVERAGE(A1,A2) which will give the average of a and b. I copied this formula down column a and when the same value kept repeating itself (technically each value is different but excel only displays a certain number of digits and they were all the same) it was clear that was the amount that we were converging on. And sure enough, for any values of a and b I experimented with, the final equation held.

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